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Non-Local Problem for Malmsteen Abstract Equation

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Abstract: The conditions concerning the unique solvability of nonlocal problems for Malmsten abstract differential equation were found. The non-local conditions contain either Erdeyi-Kober operator or Riemann-Liouville operator of fractional integration.

Key words: Nonlocal condition, the unique solvability of Malmsten equation, Erdeyi-Kober operator, Riemann-Liouville operator, Russia

INTRODUCTION

Let A is a closed operator in the Banach space E with dense determination area D(A) in E. Let's consider the differential Malmsten equation within the interval [0,1]:

$$u''(t)+\frac{k}{t}u'(t)+\frac{1}{t^2}u(t)=t^m Au(t),k,l,m\in R \quad (1)$$

Let''s seek the solution u (t)+ϵc^2 ([0,1], E) ∩ C ((0,1], D (A) of the Eq.1, satisfying nonlocal integral condition:

$$\lim_{t\to 1} I^{\beta}_{\sigma,v}u(t)=u_1 \quad (2)$$

where β>0, $I^{\beta}_{\sigma y}$ Erdeyi-Kober operator defined by the following formula (Samko *et al*., 1993; Kilbas *et al*., 2006):

$$I^{\beta}_{\sigma,v}u(t)=\frac{\sigma}{\Gamma(\beta)t^{\sigma(\beta+v)}}\int_0^t s^{\sigma v+\sigma-1}\left(t^{\sigma}-s^{\sigma}\right)^{\beta-1}u(s)ds$$

Generally speaking, the problem Eq. 1and 2 with the non-local condition in Eq. 2 is not correct. In this study they set out the conditions imposed on the operator A and the component U_1ϵE, providing its unique solubility.

Among the publications devoted to the study of nonlocal problem solution with the integral condition for abstract differential equations of the first order, let's note the publication (Tikhonov, 1998); (Sil'chenko, 2008). The criterion of uniqueness for the solution is set in (Tikhonov, 2003). Concerning the non-local problem in Eq. 1 and 2 it is considered for the first time.

Along with the Malmsten eq.1 at k>0 let's consider Euler-Poisson-Darboux equation (a special case of Malmsten equation at l = m = 0):

$$u''(t)+\frac{k}{t}u'(t)=Au(t),t\in(0,1) \quad (3)$$

According to the results of the research by (Glushak (1997) and Glushak (2016) the correct formulation of initial conditions for the Euler-Poisson-Darboux eq.3 is to set an initial value at the point t = 0:

$$u(0)=u_0\in D(A) \quad (4)$$

and the condition:

$$u'(0)=0 \quad (5)$$

which is not set (removed) at k≥1, that is typical for a number of equations with the peculiarity in the coefficients at t = 0.

The research by Glushak (1997) and Glushak (2016) also provide the conditions for the operator A to ensure that correct solution of the problem in eq.3-5. The set of operators A with which the problem in Eq.3-5 is uniformly correct, is denoted by G_k

In particular, if the operator A is limited, then A϶G_k and the problem in Eq. 3-5 has the following form:

$$u(t)=Y_k(t)u_0=\Gamma\left(\frac{k+1}{2}\right)\sum_{j=0}^{\infty}\frac{(t/2)^{2j}A^j u_0}{j!\Gamma((k+1)/2+j)}={}_0F_1\left(\frac{k+1}{2};\frac{t^2}{4}A\right)u_0,u_0\in E \quad (6)$$

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where (O) gamma function, ↓p F ↓q (O generalized hypergeometric function.u_0DA. In the case of an unbounded operator A∈G_k at the solution of the problem in Eq.3-5 has the following form (Glushak, 1997; Glushak and Pokruchin, 2016):

$$u(t) = Y_k(t)u_0 = \frac{2^{(k-1)/2\Gamma((k+1)/2)}}{i\pi(k-1)/2}\int_{\sigma-i\infty}^{\sigma+i\infty}\lambda^{(3-k)/2} \tag{7}$$
$$1_{(k-1)/2}(t\lambda)R(\lambda^2)u_0 d\lambda, \sigma > \omega,$$

where l↓v (O-modified Bessel function, λ^2 at Re$\lambda>\omega\geq0$ belongs to the resolvent set p (A) of the operator A and QUOTE R (λ^2) = (λ'2l-A)' (-1) its resolvent. The formulae Eq. 6 and 7 denote the Operator Bessel Function (OBF) resolving task operator Eq. 3-5 via Y_k (t).

The work (Glushak and Pokruchin, 2011) showed that OBF Y_k (t) may be used for the Eq.1 development of Cauchy problem weight solutions for Malstem abstract Eq. 1 (classical Malmsten equation is considered in (Watson, 1945) and the following theorem is proved.

Theorem 1: Let μ = ,u D (A) and the operator A∈ $G_{2\mu+1}$ Then the function:

$$u(t) = t(1-k+\mu(m+2)/^2{}_{Y2\mu+1}(\tau)u_{0,}\tau = \frac{2t^{(m+z)/z}}{m+2} \tag{8}$$

is the only solution for Malmsten Eq. 1 satisfying the following initial condition:

$$\lim_{t\to0} t^{(k-1-v(m+2))/2}u(t) = u_{0,} \tag{9}$$

It should be noted that under the assumptions of the theorem 1, for the considered differential Eq. 1 of the second order, as well as for Euler-Poisson-Darboux equation with k≥1, the second initial condition is not set at t = 0. Cauchy weight problem with two initial conditions will also be discussed further. The studies concerning the solubility of the problem in Eq. 1 and 2 are devoted to the initial element obtaining in the term of Eq. 9 according to non-local term in Eq. 2. At that the following function will play an important role:

$$x(k,l,m,\beta,\sigma,v;\lambda) =$$
$$\sum_{j=0}^{\infty}\frac{\Gamma(\mu+1)\Gamma(v+1((m+2)(\mu+2j)}{j!\Gamma(\mu+1+j)\Gamma(\beta+v+1+}$$
$$\frac{+1-2)/(2\sigma)(\lambda/(m+2')^{zj})}{((m+2)(\mu+2j)+1-k)/(2\sigma))}$$

which is called the characteristic function of the non-local conditions in Eq. 2 and which will be denoted x (ϖ;λ) re ϖ = (k, l, m, β, σ, v)

where

Theorem 2: Let>0, $\mu\geq0$, $\sigma>0$, $2\sigma(v+1)+\mu(m+2)+1-k>0$, A-the limited u operator and E. In order to make eq. 1 and 2 have a unique solution, it is necessary and sufficiently that the following condition was satisfied on the spectrum σ (A) of the operator A:

$$x(\varpi;\lambda) \neq 0, \lambda \in \sigma(A) \tag{10}$$

Proof: In order to find the initial element u included in the Eq. 9 for the function u (t) defined by the Eq.6 and 8 Erdeyi operator-Kober operator is used.$1^{\beta}_{\sigma,v}$ We obtain the following after elementary transformations:

$$1^{\beta}_{\sigma,v}u(t) = 1^{\beta}_{\sigma,v}t(1-k+\mu(m+2))$$
$$/^2Y^{2\mu+1}(\tau)u_0 = \left(\tau = \frac{2t^{(m+2)/2}}{m+2}\right)$$
$$= \frac{\sigma}{\Gamma(\beta)t^{\sigma}(\beta+v)}\int_0^t S^{\sigma v+\sigma-1+(1-k+\mu(m+2))/2}$$
$$(t^{\sigma}-S^{\sigma})^{\beta-1}Y_{2\mu+1}\left(\frac{2S^{(m+z)/z}}{m+2}\right)u_0 ds$$
$$= \frac{\sigma\Gamma(\mu+1)}{\Gamma(\beta)t^{\sigma(\beta+v)}}\sum_{j=0}^{\infty}\frac{(Au_o/(m+2)2)^j}{j!\Gamma(\mu+1+j)}$$
$$\int_0^t S^{\sigma v+\sigma-1+(1-k+\mu(m+2))/2}(t^{\sigma}-S^{\sigma})^{\beta-1}ds$$

Calculating the last integral, we obtain the following equation by the virtue of the condition of Eq. 2:

$$\sum_{j=0}^{\infty}\frac{\Gamma(\mu+1)\Gamma(v+1+((m+2)(\mu+2j)+1-k)/}{j!\Gamma(\mu+1+j)\Gamma(\beta+v+1+(m+2)} = u_1$$
$$\frac{(2\sigma))(Au_o/(m+2)^j)}{(\mu+2j)+1-k)/(2\sigma))} \tag{11}$$

Let Ω is an open set of a complex plane containing the spectrum σ (A) of the limiting operator A, the boundary of which $\partial\Omega$ consists of a finite number of rectifiable Jordan curves, oriented in a positive direction. Then, putting down the representation via the resolvent for the operator on the left side of Eq.11, let's rewrite of Eq.11 in the following form:

$$Bu_0 \equiv \int_{\partial\Omega} x(\varpi;\lambda) R(\lambda) u_0 d\lambda = u_1 \quad (12)$$

Therefore, a necessary and sufficient condition for the unique solvability of the problem in Eq. 1 and 2 with the bounded operator A is the solvability of the Eq. 12, i.e., the absence of the point $\lambda = 0$ in the operator B spectrum σ (B). The Eq. 12 means that the operator B is an analytic function of the operator A,B = x (ϖ;A) . According to the theorem about bounded operator range reflection. $\sigma(B) = x(\varpi;\sigma(A))$. Thus, the value $\lambda = 0$ is not the operator B spectrum point only when, it is not turned into zero function x ($\varpi;\lambda$) within the spectrum σ (A) or when the condition in Eq. 10 is performed which is the same.

When the condition in Eq. 10 is performed the initial element $u_0 = B^{-1} u_1$ and the solution of the problem in Eq. 1 and 2 with the limited operator is defined by the in Eq. 8 and 6. The theorem is proved. In the case when the parameters σ and v are chosen in a special way, i.e.: $\sigma = m + 2$ and:

$$v = \mu/2 + (k-1)/(2m+4) \text{and}^z = 2\sqrt{\lambda/(m+2)}$$

then:

$$x(\varpi;\lambda) = \frac{\Gamma(\mu+1)}{\Gamma(\beta+\mu+1)}\, {}_0F_1\left(\beta+\mu+1; \frac{z^2}{4}\right) = \Gamma(\mu+1)\left(\frac{2}{z}\right)^{\beta+\mu} l_{\beta+\mu}(z)$$

where $l_{\beta+\mu}(z)$ modified Bessel function and the statement is true.

Theorem 3: Let $\mu \geq 0$, $\sigma = m+2$, $v = \mu/2+(k-1)/(2m+4)$, A-limited operator, u_1. In order to make the non-local problem (1), (2) have a unique solution at the specified parameter values, it is necessary and sufficient that the following condition was satisfied on the spectrum $\sigma(A)$ of the operator A:

$$l_{\beta+\mu}\left(\frac{2\sqrt{\lambda}}{m+2}\right) \neq 0, \lambda \in \sigma(A) \quad (13)$$

If the conditions of the Theorem 3 are satisfied and $k \geq 1$,1 = m = 0, then the non-local problem in Eq 1 and 2 for Malmsten equation turns into a non-local problem for the Euler-Poisson-Darboux equation, which was studied in (Glushak, 2016) and the corresponding characteristic function in this case has the following form:

$$\chi(\varpi;\lambda) = \Gamma\left(\frac{k+1}{2}\right)\left(\frac{2}{\sqrt{\lambda}}\right)^{(k-1)/2+\beta} l_{(k-1)/2+\beta}\left(\sqrt{\lambda}\right)$$

Therefore, the distribution and the asymptotic behavior of the zeros $\lambda_j = \lambda_j$ (k-1)/2+β j = 1,2,... the functions x ($\varpi;\lambda$) are known (Watson, 1945), chapter 15), i.e.: all zeros are simple, negative and:

$$\lim_{j\to\infty} \frac{\lambda_j}{j^2} = -\pi^2 \quad (14)$$

According to the proved Theorem 2 it follows that the location of the function x ($\varpi;\lambda$) zeros determines the unique solvability of the problem in Eq. 1 and 2 with the limited operator . For the Eq. 1 with the unlimited operator A the condition of the form of Eq.10 is no longer sufficient for unique solvability, though the location of zeros also plays an important role.

Let's establish then the necessary condition for the uniqueness of the inverse problem in Eq.1 and 2 solution with an unbounded operator A.

Theorem 4: Let $\mu \geq 0$, A the closed linear operator in E. Suppose that the non-local problem in Eq. 1 and 2 has the solution u (t). In order to make this solution the only one it is necessary to ensure that no zero λ_j, j = 1, 2...of the whole function x ($\varpi;\lambda$) is the own value of the operator A. Proof. Suppose that some zero λ_0 from the countable set of function zeros x ($\varpi;\lambda$) is the eigenvalue of the operator A with the eigenvector $h_0 = 0$. A direct verification shows that the function:

$$\vartheta(t;\lambda_0) = (1-k+\mu(m+2))/^2\, Y_{2\mu+1}(\tau) = t^{(1-k+\mu(m+2))/2}\, {}_0F_1\left(\mu+1; \frac{\lambda_0\tau^2}{4}\right), \tau = \frac{2t^{(m+2)/2}}{m+2}$$

Is the scalar problem solution:

$$\vartheta(t;\lambda_0) + \frac{k}{t}\vartheta'(t;\lambda_0) + \frac{1}{t^2}\vartheta(t;\lambda_0) = \lambda_0 t^m \vartheta(t;\lambda_0), t \in (0,1],$$

$$\lim_{t\to 1} l^{\beta}_{\sigma,v}\vartheta(t;\lambda_0) = 0.$$

We note only that there is zero nonlocal condition as λ_0 the function zero x ($\varpi;\lambda$). Thus, the function u (t) = v (t;λ_0) h_0 is a non-zero solution of the homogeneous (u_1=0) non-local problem in Eq. 1 and 2. Thus the solution of the problem in Eq. 1 and 2 is clearly not unique, if it exists. The theorem is proved.

Now let's turn to sufficient condition for unique solvability of the problem in Eq. 1 and 2 with an unbounded operator A. As well as during the proof of the theorem 2, we have to find the initial element from the following equation:

$$\lim_{t\to 1} l^{\beta}_{\sigma,v} t\left(1-k+\mu(m+2)\right)/^{2}\ Y_{2\mu+1}(\tau)u_0 = u_1, \tau = \frac{2t^{(m+2)/2}}{m+2} \tag{15}$$

Where OBF $Y_{2\mu+11}$ (τ) is determined by the Eq. 7. Let's demonstrate the left side of the Eq. 15 in the following form after elementary transformations:

$$\begin{aligned}
&\lim_{t\to 1} l^{\beta}_{\sigma,v} t\left(1-k+\mu(m+2)\right)/^{2}\ Y_{2\mu+1}(\tau)u_0 = \frac{\sigma}{\Gamma(\beta)}\int_0^1 s^{\sigma v+\sigma-1}\left(1-s^{\sigma}\right)^{\beta-1} \times \frac{2^{\mu}\Gamma(\mu+1)}{i\mu s^{\mu}}\int_{\sigma+i\infty}^{\sigma+i\infty} \lambda^{1-\mu} l_{\mu}(\xi\lambda)R\left(\lambda^2\right) \\
&u_0 d\lambda ds = \left(\xi = \frac{2s(m+2)/2}{m+2}\right) = \frac{\sigma\left(m+2\mu\Gamma(\mu+1)\right)}{i\mu\Gamma(\beta)}\int_{\sigma+i\infty}^{\sigma+i\infty} \lambda^{1-\mu} l_{\mu}(\xi\lambda) \\
&R\left(\lambda^2\right)u_0 d\lambda ds = \left(\xi = \frac{2s^{(m+2)/2}}{m+2}\right)\frac{\sigma\left(m+2\mu\Gamma(\mu+1)\right)}{i\mu\Gamma(\beta)}\int_{\sigma+i\infty}^{\sigma+i\infty} \lambda^{1-\mu}R\left(\lambda^2\right)u_0 \times \int_0^1 S^{\sigma v+\sigma-k/2}\left(1-s^{\sigma}\right)^{\beta-1} \\
&\sum_{j=0}^{\infty}\frac{\left(\lambda sm/2+1/(m+2)^2\right)}{j!\Gamma(\mu+1+j)} ds d\lambda = \frac{1}{i\mu}\int_{\sigma+i\infty}^{\sigma+i\infty} \lambda R\left(\lambda^2\right)u_0 \\
&\sum_{j=0}^{\infty}\frac{\Gamma(\mu+1)\Gamma\left(v+1\left(\begin{matrix}(m+2)(\mu+2j)+\\ 1-k)/\left(2\sigma(\lambda/(m+2)\right)^{2j}\end{matrix}\right)\right)}{j!\Gamma(\mu+1+j)\Gamma\left(\beta+v+1+\left(\begin{matrix}(m+2)\\ (\mu+2j)+1-k\end{matrix}\right)/2\sigma\right)} d\lambda = \frac{1}{i\mu}\int_{\sigma+i\infty}^{\sigma+i\infty} x\left(\varpi;\lambda^2\right)\lambda R\left(\lambda^2\right)u_0 d\lambda
\end{aligned} \tag{16}$$

Taking into account the representation Eq. 16, let put down the Eq. 15 in the following form:

$$Bu_0 \equiv \frac{1}{i\mu}\int_{\sigma+i\infty}^{\sigma+i\infty} x\left(\varpi;\lambda^2\right)\lambda R\left(\lambda^2\right)u_0 d\lambda = u1 \tag{17}$$

Thus, the unique solvability of the problem Eq. 1 and 2 reduces to the existence problem of the bounded operator, Eq. 17 set by the relation Eq. 17 and continued by continuity on E, an inverse operator defined according to some subset D (A).

As we noted earlier, during the proof of a sufficient condition for the unique solvability of the problem Eq. 1 and 2 with an unbounded operator a crucial role is played by the distribution and the asymptotic behavior of the function zeros x ($\varpi;\lambda$). Therefore, we consider a special case first when the distribution and the asymptotic behavior of the function zeros x ($\varpi;\lambda$) are known.

Condition 1: Let ≥ 0, σ = m+2,v = μ/2+k-1/2m+4 and each λ_j, j = 1, 2...zero of the function:

$$x(\varpi;\lambda) = \Gamma(\mu+1)\left(\frac{m+2}{\sqrt{\lambda}}\right)^{\beta+\mu} l_{\beta+\mu}\left(\frac{2\sqrt{\lambda}}{m+2}\right) \tag{18}$$

belongs to the resolvent set p (A) of operator A and such d>0 exists, that:

$$\sup_{j=1,2,\ldots}\left\|R\left(\lambda_j\right)\right\| \leq d$$

Let's assume that the condition 1 is fulfilled. Since each zero λ_j, j = 1,2,... defined by the Eq. 18 of the function x ($\varpi;\lambda$) belongs to p(A), then it belongs to p (A)together with the circular neighborhood Ω_j of the radius 1/d, the boundary of which, passed along clockwise is denoted as λ_j. Let λ_0 is the complex plane contour, consisting of a straight line passable passable bottom up Re z = $\sigma_0>\omega$, γ_0^2-parabola, γ_0 image at the following representation:

$$\lambda_0 \in p(A), \mathrm{Re}\lambda_0 > \sigma > \sigma_0$$

Let's take and choose n∈N so that:

$$n > \max\left\{\begin{matrix}(k+2\beta+2\mu+3)\\ /4,(\beta+\mu+5/2)/2\end{matrix}\right\} \tag{19}$$

Let us consider the bounded operator:

$$H_v = \frac{1}{2\mu i}\int_{\Xi}\frac{R(z)vdz}{x(\varpi;z)(z-\lambda_0)^{n'}}H: E\to E \tag{20}$$

Let us show that the integral in Eq. 20 at the performance of condition 1 is absolutely convergent.

Indeed, due to the choice of the circuit ,γ_0^2 the inequalities (Glushak and Pokruchin, 1997; Glushak and Pokruchin, 2016):

$$\left\|\lambda^{1-k/2}\left(\lambda^2\right)\right\| \le \frac{M}{\left(\mathrm{Re}\lambda-\omega\right)^{k/2+1}} \mathrm{Re}\lambda > \omega$$

and the asymptotic behavior of the modified Bessel function:

$$1_v(\lambda) = \frac{e^{\lambda}}{\sqrt{2\mu\lambda}}\left(1+o\left(\lambda^{-1}\right)\right)$$
$$,\lambda \to \infty |\arg\lambda| < \mu/2$$

the integral is absolutely convergent, since, as follows from (19), 2n> (k+1)/2+β+μ+1 Let's consider the integral according to $\bigcup_{j=1,2,...} \gamma_j$. We obtain the following:

$$\frac{1}{2\pi i}\int_{\bigcup_{j=1,2,...}\gamma_j} \frac{R(z)dz}{\chi(\varpi;z)(z-\lambda_0)^n} = -\sum_{j=1}^{\infty}\frac{R(\lambda_j)}{\chi'(\varpi;\lambda_j)(\lambda_j-\lambda_0)^n} =$$

$$= -\frac{(m+2)^{1-\beta-\mu}}{?\,(\mu+1)}\sum_{j=1}^{\infty}\frac{\lambda_j^{(\beta+\mu)/2+3/4}R(\lambda_j)}{\lambda_j^{1/4}I_{\beta+\mu+1}\left(2\sqrt{\lambda_j}/(m+2)\right)(\lambda_j-\lambda_0)^n}$$

and the absolute convergence of the obtained series follows from the condition 1, the asymptotic behavior of the modified Bessel function and the asymptotic behavior in Eq. 14 of the modified Bessel function zeros, since as follows from Eq. 19, 2n>β+μ+5/2

Theorem 5: Suppose that the condition 1 and A ϵ $G_{2\mu+1}$ performed. If u_1DA+_1 where nϵN is chosen so that the inequality Eq. 19 is performed, then the problem Eq. 1 and 2 has a unique solution. Proof. As we noted already, the only existing solution to the problem Eq. 1 and 2 is reduced to the existence of an inverse element operator among the limited operator B, defined by the equation Eq. 17 and 18. Let us show that the inverse operator B has an inverse operator . Let's. V ϵ D (A), $\sigma_0<\sigma<\mathrm{Re}\xi$ Then, substituting the operator B in (20) defined by the equality in Eq. 17 and applying the Hilbert identity:

$$R(z)R(\xi^2) = \frac{R(z)-R(\xi^2)}{\xi^2-z}$$

we obtain the following equality:

$$\mathrm{HBv} = \frac{1}{2\mu i}\int_{\Xi}\frac{\frac{R(z)}{x(\varpi;z)(z-\lambda_0)^n}\frac{1}{in}}{\int_{\sigma+i\infty}^{\sigma+i\infty}x(\varpi!\xi^2)\xi R(\xi^2)vd\xi dz} =$$
$$= -\frac{1}{2\mu^2}\int_{\Xi}\int_{\sigma+i\infty}^{\sigma+i\infty}\left(\frac{\xi x(\varpi;\xi^2)R(z)_v}{x(\varpi;z)(z-\lambda_0)^n(\xi^2-z)} - \frac{\xi x(\varpi;\xi^2)R(z)_v}{x(\varpi;z)(z-\lambda_0)^n(\xi^2-z)}\right) \quad (21)$$

The integral converges absolutely in Eq. 21. Changing the order of integration, we will have the following:

$$\mathrm{HBv} = -\frac{1}{2\pi^2}\int_{\mathfrak{Z}}\int_{\sigma-j\infty}^{\sigma+j\infty}\frac{\xi\chi(\varpi;\xi^2)R(z)vd\xi dz}{\chi(\varpi;z)(z-\lambda_0)^n(\xi^2-z)}$$
$$+\frac{1}{2\pi^2}\int_{\sigma-j\infty}^{\sigma+j\infty}\xi\chi(\varpi;\xi^2)R(\xi^2)v\int_{\Xi} \quad (22)$$
$$\frac{dz}{\chi(\varpi;z)(z-\lambda_0)^n(\xi^2-z)}d\xi$$

If the integration contour γ_0^2 is closed to the left without crossing $\bigcup_{j=1,2}\gamma_j$, then the inner integral in the second term of in Eq. 22 will turn into zero because of the circuit choice Ξ and Cauchy's theorem for multiply connected domain. And, Let's use Cauchy's integral formula to calculate the integrals in the first term of Eq. 22. Thus, the following equality is fair:

$$\mathrm{HBv} = -\frac{1}{2\pi^2}\int_{\Xi}\int_{\Upsilon}\frac{\xi\chi(\varpi;\xi^2)R(z)vd\xi dz}{\chi(\varpi;z)(z-\lambda_0)^n(\xi^2-z)}$$
$$= -\frac{1}{4\pi^2}\int_{\Xi}\int_{\Upsilon 2}\frac{\xi\chi(\varpi;\lambda)R(z)vd\lambda dz}{\chi(\varpi;z)(z-\lambda_0)^n(\lambda-z)} = \frac{1}{2_{\pi i}}\int_{\Xi}$$
$$\frac{R(z)vdz}{(z-\lambda_0)^n} = \frac{1}{2_{\pi i}}\int_{\gamma_0^2}\frac{R(z)vdz}{(z-\lambda_0)_n} =$$
$$= \frac{-1}{(n-1)!}R^{(n-1)}\lambda_0 v = (-1)^n R^n(\lambda_0)v$$

The commuting operators H, B, $R^n(\lambda_0)$ are limited and the domainis D (A) is dense in E, thus, the equality Hbv = $(-1)^n(\lambda_0)^v$ is also fair for v ϵ E. At that HB: E→D (A^n). Thus, the operator $B^{-1}v = (-1)^n(\lambda_0 I-A)^n Hv$ at v ϵ is the inverse one with regard to B, B^{-1}:D (A^n→E). Indeed, $BB^{-1}v = (-1)^n B(\lambda_0 I-A)^n Hv = (-1)^n BH(\lambda_0 I-A)^n v = R^n(\lambda_0)(\lambda_0 I-A)^n v = v$, vϵ D ($A^n$), $B^{-1}Bv = (-1)^n(\lambda_0 I-A)^n R^n(\lambda_0)v = v$ϵE.

Returning to the problem in Eq. 1 and 2, let's define the initial element u_0 (λ I_0-A)^n H $_{u1}$belonging to D (A), where u_1 ∈D (A^{n+1}),The operator H is set by the equality in Eq. 20, λ_0 ∈ p (A), Reλ_0>σ_0>ω.Then the only correct solution u (t) of the problem in Eq. 1 and 2 has the following form:

$$u(t) = t^{(1-k+\mu(m+2))/2} Y_{2\mu+1}(\tau)u_0, \tau = \frac{2t^{(m+2)/2}}{m+2}$$

where OBF $Y2_{\mu+1}$ (τ)is defined by the Eq. 7. The theorem is proved. In contrast to the theorem 5, the location, multiplicity, and the asymptotic behavior of the characteristic function λ (ϖ;λ) zeros are unknown in general case, which leads to additional difficulties to prove the sufficient conditions for the unique solvability of the problem in Eq 1 and 2 with an unbounded operator A.

Condition 2: Let μ≥0 and each path λ_j, j = 1,2,... the function x (ϖ;λ) belongs to the resolvent set p (A) of the operator A and such d>0 $_{exists}$, that:

$$\sup_{j=1,2,...} \|R(\lambda_j)\| \le d$$

Besides, let n∈N, λ_0 ∈ p (A), Re λ_0>$\sigma_0$are chosen so that the integral is completely converged in the form of in Eq. 20. Similarly, the following statement is proved in theorem 5.

Theorem 6: Suppose that condition 2 and A ∈ $G_{2\mu+1}$are performed. If u_1 ∈D (A^{n+1}), then the problem in Eq. 1 and 2 has the only solution. The condition 2 was formulated in a very general way. The additional information about the zeros of the characteristic function, x (ϖ;λ) will allow to specify it as well as during the formulation of the condition 1.

MATERIALS AND METHODS

Malmsten equation makes possible (Glushak and Pokruchin, 2011) one more correct statement of the weight Cauchy problem, but with two initial conditions, in contrast to the first weight problem Eq. 1 and 2 At 0 <μ≤ 1/2 we will seek the solution of the Eq. 1 satisfying the following conditions:

$$\lim_{t\to 0}(t^{(k-1+\mu(m+2))/2}u(t)) = u_0, \qquad (23)$$

$$\lim_{t\to 0} t^{-m/2}(t^{(k-1+\mu(m+2))/2}u(t))' = 0 \qquad (24)$$

OBF y_k (t) is also used (Glushak and pokruchin, 2011) for the development of the problem in Eq. 1 and 23 solution.

Theorem 7: Let0<μ≤1/2,u_0 ∈D (A) and the operator A∈ $G_{1-2\mu}$. Then the function:

$$u(t) = t^{(1-k-\mu(m+2))/2} Y_{1-2\mu}(\tau)u_0, \tau = \frac{2t^{(m+2)/2}}{m+2}$$

Is the only solution of the problem in Eq 1, 23 and 24. In particular, at l = m = 0, 0<k<1 the problem Eq 1, 23 and 24 turns into the problem in Eq. 3-5 for Euler-Poisson-Darboux equation. At that the initial condition in Eq. 5 is not removed.

And in the present case of the study concerning the solvability of a nonlocal problem in Eq. 1, 2 and 24 they are also dedicated to the obtaining of the primary element in condition in Eq. 23 according to non-local condition Eq. 2 The meaning of the statements cited below is to define by the means of non-local condition Eq. 2 characteristic features which will be denoted by us as follows :

$$\tilde{\chi}(\varpi;\lambda) = \sum_{j=0}^{\infty} \frac{\Gamma(1-\mu)\Gamma(v+1+((m+2)(2j-\mu)}{j!\Gamma(1-\mu+j)\Gamma(\beta+v+1+((m+2)} \frac{+1-k)/(2\sigma))(\lambda/(m+2)^2)^{2j}}{(2j-\mu)+1-k)/(2\sigma))}$$

to obtain the sufficient conditions for the solvability of the nonlocal problem in Eq.1, 2 and24. Similarly, the following theorems are proved according to Theorems 2-6.

Theorem 8: Let, β>0,0<μ≤1/2, σ>0,2σ (v+1)-μ (m+2)+1-k0, A is the limited operator and u_1 ∈ E. Suppose, that the problems in Eq. 1, 2and 24 have a unique solution. It is necessary and sufficient that the following term is performed x (ϖ;λ) = 0, λ ∈ σ (A) in the range σ (A) of the operator A.

Theorem 9: Let,0<?0≤1/2, σ = m+2,v = μ/2+(k-1)/(2m+4), A,-the limited operator, u_l ∈ E In order to make the nonlocal problems Eq. 1, 2 and 24 have a unique solution at the specified values of the parameters it is necessary and sufficient that the following condition is performed on the spectrum σ (A)of the operator A:

$$I_{\beta-\mu}\left(\frac{2\sqrt{\lambda}}{m+2}\right) \neq 0, \lambda \in \sigma(A)$$

If the conditions of Theorem 9 are performed and, besides, 0<k<I, L=m = 0, then the nonlocal task Eq. 1, 2and 24 for Malmsten equation is turned into the non-local

problem Euler-for Poisson-Darboux equation and the corresponding characteristic function in this case has the following form :

$$\tilde{\chi}(\varpi;\lambda)=\Gamma\left(\frac{k+1}{2}\right)\left(\frac{2}{\sqrt{\lambda}}\right)^{(k-1)/2+\beta} I_{(k-1)/2+\beta}\left(\sqrt{\lambda}\right)$$

Theorem 10: Suppose $0 < \mu \le 1/2$, is a closed linear operator in E Let the nonlocal problem Eq. 1, 2 and 24 can be solved by u (t). In order to make the solution the only one it is necessary that neither zero, λ_j, j = 1,2,...of the whole x (ϖ;λ) function is the eigenvalue of the operator A.

Condition 3: Let $0<\mu\le 1/2$, $\sigma = m+2$, $v=-\mu/2+(k-1)/(2m+4)$ and each zero λj, j = 1, 2, ... of the function:

$$\tilde{\chi}(\varpi;\lambda)=\Gamma(1-\mu)\left(\frac{m+2}{\sqrt{\lambda}}\right)^{\beta-\mu} I_{\beta-\mu}\left(\frac{2\sqrt{\lambda}}{m+2}\right)$$

belongs to the resolvent set $\rho(A)$ of the operator and there is such d> 0,that:

$$\sup_{j=1,2,...}\left\|R(\lambda_j)\right\|\le d$$

Let's take $\lambda_0 \in \rho(A)$, Re $\lambda_0 > \sigma > \sigma_0$ and let's $n\in$ N choose so that:

$$n>\max\left\{(k+2\beta-2\mu+3)/4,\ (\beta-\mu+5/2)/2\right\} \quad (25)$$

Theorem 11: Suppose that the condition 3 and $A\in G_{1-2\mu}$ are fulfilled: If $u_1 \in D(A^{n+1})$ where $n\in N$ is chosen in such a way, that the Eq. 25 is performed, then the problem Eq. 1, 2 and 24 has a unique solution.

Condition 4: Let $0<\mu\le 1/2$, and each zero λj, j = 1, 2 of the function $\tilde{\chi}(\varpi;\lambda)$ belongs to the resolvent set $\rho(A)$ of the operatorand such d> 0 exists that:

$$\sup_{j=1,2,...}\left\|R(\lambda_j)\right\|\le d.$$

Let's $n\in N$, $\lambda_0 \in \rho(A)$, Re $\lambda_0>\sigma_0$ are chosen so, that the integral is converged completely:

$$\int_{\Xi}\frac{R(z)v\,dz}{\tilde{\chi}(\varpi;z)(z-\lambda_0)^n}$$

Theorem 12: Let the condition 4 and $A\in G_{1-2\mu}$ are fulfilled. If $u_1 \in D(A_{n+1})$, then the problem Eq. 1, 2 and 24 has the unique solution.

RESULTS AND DISCUSSION

Let's consider then the problem of finding Malmsten Eq. 1 solutions satisfying the local condition:

$$\lim_{t\to 1} I^{\beta}u(t)=u_2, \quad (26)$$

where I^{β}, $\beta>0$ the left-sided fractional Riemann-Liouville integral, defined by the equation: (Samko *et al*., 1993)

$$I^{\beta}u(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(s)\,ds.$$

Let us determine further the initial element u_0 in term Eq. 9 according to non-local condition in Eq. 26. To this end let introduce the following function for consideration:

$$\psi(k,l,m,\beta;\lambda)=\sum_{j=0}^{\infty}\frac{\Gamma(\mu+1)\Gamma(((m+2)(\mu+2j)+}{j!\Gamma(\mu+1+j)\Gamma(\beta+((m+2)},\ \frac{3-k)/2)\ (\lambda/(m+2)^2)^{2j}}{(\mu+2j)+3-k)/2)}$$

which is called the characteristic function of the non-local condition in Eq. 26 and which will be denoted as $\psi(\varpi;\lambda)$, where ϖ = (k, l, m, β).

Theorem 13: Let $\beta>0$, $\mu\ge 0$, A-the limited operator and $u_2\in E$. In order to make the problems in Eq. 1 and 26 have a unique solution it is necessary and enough that the condition $\psi(\varpi;\lambda)\ne 0$, $\lambda\in\sigma(A)$ is fulfilled within the range σ (A) of the operator A.

Proof. In order to find the initial element u_0 included in the Eq. 9 determined by the Eq. 8 and 6 of the function u(t) let's use Riemann-Liouville operator $I\beta$. We will obtain the following after elementary transformations:

$$I^{\beta}u(t)=I^{\beta}t^{(1-k+\mu(m+2))/2}Y_{2\mu+1}(\tau)u_0=\left(\tau=\frac{2t^{(m+2)/2}}{m+2}\right)$$

$$=\frac{\Gamma(\mu+1)}{\Gamma(\beta)}\sum_{j=0}^{\infty}\frac{\left(Au_0/(m+2)^2\right)^j}{j!\,\Gamma(\mu+1+j)}\int_0^t s^{((m+2)(\mu+2j)+1-k)/2}(t-s)^{\beta-1}ds.$$

Calculating the last integral, by the virtue of condition in Eq. 26, we obtain the following equation:

$$\sum_{j=0}^{\infty}\frac{\Gamma(\mu+1)\Gamma(((m+2)(\mu+2j)+3-k)/2)}{j!\Gamma(\mu+1+j)\Gamma(\beta+((m+2)(\mu+2j)+3-k)/2)}\left(Au_0/(m+2)^2\right)^{2j}=u_2 \quad (27)$$

Let Ω is an open set of the complex plane containing the range $\sigma(A)$ of the bounded operator A, the boundary of which $\partial\Omega$ consists of a finite number of rectifiable Jordan curves, oriented in the positive direction. Then, putting down the representation through the resolvent for the operator on the left side in eq. 27, we rewrite the Eq. 27 in the following form:

$$B_1u_0 \equiv \int_{\partial\Omega}\psi(\varpi;\lambda)R(\lambda)u_0\,d\lambda = u_2.$$

The proof similar to the Theorem 2 proof is completed. If the theorem 13 conditions are fulfilled and $k\geq 1$, $l = m = 0$, then the nonlocal problem in Eq. 1 and 26 for Malmsten equation turns into a non-local problem for Euler-Poisson-Darboux equation and the corresponding characteristic function in this case has the following form The following statement is proved similarly to theorem 4.

Theorem 14: Let A is a closed linear operator in . Suppose that the non-local problem in Eq. 1 and 26 has the solution . In order to make this solution a unique one it is necessary that no zero λj, j = 1, 2,... of the whole function $\psi(\varpi;\lambda)$ is not an eigenvalue of the operator A.

Condition 5: Let $\mu\geq 0$ and each zero λj, j = 1, 2, ... of the function $\psi(\varpi;\lambda)$ belongs to the resolvent set $\rho(A)$ of the operator A and there is such d>0, that:

$$\sup_{j=1,2,\ldots}\|R(\lambda_j)\|\leq d.$$

Besides, let $n\in N$, $\lambda_0\in\rho(A)$, Re $\lambda_0>\sigma_0$ are chosen so, that the following integral was converged completely:

$$\int_{\Xi}\frac{R(z)vdz}{\psi(\varpi;z)(z-\lambda_0)^n}$$

The following statement is proved like theorem 5.

Theorem 15: Let the condition 5 and $A\in G_{2\mu+1}$ are performed. If $u_2\in D(A^{n+1})$, then the problem in Eq. 1 and 26 has a unique solution.

CONCLUSION

This problem is solved by the function obtaining which satisfies Malmsten Eq. 1, the non-local condition in Eq. 26 and the weight original condition in Eq. 24. It is assumed, that $0<\mu\leq 1/2$ and $A\in G_{1-2\mu}$.

The statements similar with the theorems 13-15 are fair for the problems (1), (24), (26). At that instead of the function $\psi(\varpi;\lambda)$ one should use the following function:

$$\tilde{\psi}(\varpi;\lambda)=\sum_{j=0}^{\infty}\frac{\Gamma(1-\mu)\Gamma(((m+2)(2j-\mu)+3-k)/2)\left(\lambda/(m+2)^2\right)^{2j}}{j!\Gamma(1-\mu+j)\Gamma(\beta+((m+2)(2j-\mu)+3-k)/2)}$$

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