Journal of Engineering and Applied Sciences 14 (22): 8312-8314, 2019
ISSN: 1816-949X

On the n-Normed Space of Continuous Functions

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Abstract: In Sixties the concept of 2-normed spaces was initially developed by Gahler while that of n-normed space one can see in Misiak. Since, then many others have studied this concepts and obtained various results. Mutaqin and Gunawan studied the relation between two known n-norms on l^p, the space of p-summable sequences. The purpose of this study is to study the relation between the two n-norms on L_∞, the space of all continuous functions. The first n-norm is taken from Gunawan definition while the second n-norm is derived from Gahler's formula. In particular, we examine the convergence in terms of these n-norms and prove that the convergence in terms of each of these n-norms is equivalent to that in the usual norm on L_∞.

Key words: n-normed space, dual space, convergence space, usual norm, particular, p-summable

INTRODUCTION

Let, n be a non-negative integer and X be a real vector space of dimension at least n. A real-valued function ||, ..., || on X^n satisfying the following four properties:

- $\|x_1, ..., x_n\| = 0 \leftrightarrow x_1, ..., x_n$ are linearly dependent
- $\|x_1, ..., x_n\|$ is invariant under permutation
- $\|\alpha x_1, x_2, ..., x_n\| = |\alpha|\, \|x_1, x_2, ..., x_n\|$ for any $\alpha \in \mathbb{R}$
- $\|x+x^*, x_2, ..., x_n\| \leq \|x, x_2, ..., x_n\| + \|x^*, x_2, ..., x_n\|$

where, an n-norm on X and the pair (X, ||, ..., ||) is called an n-normed space. A sequence f_m in an n-normed space (X, ||, ..., ||) is said to converge to some $f \in X$ in the n-norm whenever, $\lim_{m\to\infty} \|f_m+f, f_2, ..., f_n\| = 0$, for every $f_2, ..., f_n \in X$. On the space L_∞, the following an n-norm was defined by Gunawan (2001):

$$\|g_1, ..., g_n\|_\infty = \sup_{x_1} . \sup_{x_2}, ..., \sup_{x_n} \left|\det\left(g_i\left(x_j\right)\right)\right|$$

The theory of n-normed spaces was developed by Gahler (1969a-c). While various aspects of n-normed spaces have been studied extensively (Jain and Chugh, 1995; Kim and Cho, 1996; Malceski, 1997; Mutaqin and Gunawan, 2010; Siddiqi *et al*., 1989; Suyalatu, 1990).

If X is equipped with a norm ||. ||, then according to, Gahler, one may define an n-norm on X (assuming that X is at least n-dimensional) by the formula:

$$\|x_1, ..., x_n\|^* = \sup_{\substack{f_i \in X', \|f_i\| \leq 1 \\ i=1, ..., n}} \begin{vmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{vmatrix}$$

where, X' denotes the dual of X which consists of bounded linear functionals on X. For $X = L_\infty$, we know that $X' = L_1$. In this case the above formula reduces to:

$$\|p_1, ..., p_n\|_\infty^* = \sup_{\substack{h_i \in L_1, \|h_i\| \leq 1 \\ i=1, ..., n}} . \sup_{x_{r1}} . \sup_{x_{rn}}$$

$$\begin{vmatrix} p_1(x_{r1}) & \cdots & p_1(x_{rn}) \\ \vdots & \ddots & \vdots \\ p_n(x_{r1}) & \cdots & p_n(x_{rn}) \end{vmatrix} \begin{vmatrix} h_1(x_{r1}) & \cdots & h_1(x_{rn}) \\ \vdots & \ddots & \vdots \\ h_n(x_{r1}) & \cdots & h_n(x_{rn}) \end{vmatrix}$$

where, $p_1, ..., p_n \in L_\infty$ and $\|.\|_1$ denotes the usual norm on L_1. Thus, on L_∞, we have two definitions of n-norms, one referable to Gunawan (2001) and the other is derived from Gahler (1969a-c). Beginning, we prove the results for n = 2 and then extend it to any $n \geq 2$.

MATERIALS AND METHODS

Recall that Gunawan's definition of 2-norm on L_∞ is given by:

$$\|p, q\|_\infty = \sup_{x_r} . \sup_{x_k} \left\{ \text{abs} \begin{vmatrix} p(x_r) & p(x_k) \\ q(x_r) & q(x_k) \end{vmatrix} \right\}$$

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By the same consist as Gunawan (2001), we can get:

$$\|p,q\|_\infty^* = \sup_{h,e\in L_1\|e\|_1,\|h\|_1\le 1}.\sup_{x_r}.\sup_{x_k}\begin{vmatrix}p(x_r) & p(x_k)\\ q(x_r) & q(x_k)\end{vmatrix}.\begin{vmatrix}h(x_r) & h(x_k)\\ e(x_r) & e(x_k)\end{vmatrix}$$

From the last formula, we prove the following facts.

Fact 2.1: The inequality $\|p, q\|_\infty \le 2\|p\|_\infty.\|q\|_\infty$ holds for every $p, q \in L_\infty$.

Proof: By triangle inequality for real number and Minkowski's inequality, we have:

$$\begin{aligned}\|p,q\|_\infty &= \sup_{x_r}.\sup_{x_k}\left|p(x_r).q(x_k)-p(x_k)q(x_r)\right| \le \\ &\sup_{x_r}.\sup_{x_k}\left[\left|p(x_r)\right|\left|q(x_k)\right|+\left|p(x_k)\right|\left|q(x_r)\right|\right]\le \\ &\left[\sup_{x_r}.\sup_{x_k}\left|p(x_r)\right|\left|q(x_k)\right|+\sup_{x_r}.\sup_{x_k}\left|p(x_k)\right|\left|q(x_r)\right|\right]= \\ &2\|p\|_\infty.\|q\|_\infty\end{aligned}$$

This proves that $\|p, q\|_\infty \le 2\|p\|_\infty.\|q\|_\infty$.

Fact 2.2: The inequality $\|p, q\|_\infty^* \le 2\|p, q\|_\infty$ holds for every $p, q \in L_\infty$.

Proof: By Holder's inequality ,we have:

$$\sup_{x_r}.\sup_{x_k}\begin{vmatrix}p(x_r) & p(x_k)\\ q(x_r) & q(x_k)\end{vmatrix}.\begin{vmatrix}h(x_r) & h(x_k)\\ e(x_r) & e(x_k)\end{vmatrix}\le$$

$$\sup_{x_r}.\sup_{x_k}\left\{abs\begin{vmatrix}p(x_r) & p(x_k)\\ q(x_r) & q(x_k)\end{vmatrix}\right\}\times\sup_{x_r}.\sup_{x_k}\left\{abs\begin{vmatrix}h(x_r) & h(x_k)\\ e(x_r) & e(x_k)\end{vmatrix}\right\}$$

Now:

$$\begin{aligned}&\sup_{x_r}.\sup_{x_k}\left\{abs\begin{vmatrix}h(x_r) & h(x_k)\\ e(x_r) & e(x_k)\end{vmatrix}\right\}\le \\ &\sup_{x_r}.\sup_{x_k}\left[\left|h(x_r).e(x_k)\right|+\left|h(x_k).e(x_r)\right|\right]\le \\ &\sup_{x_r}.\sup_{x_k}\left|h(x_r).e(x_k)\right|+\sup_{x_r}.\sup_{x_k}\left|h(x_k).e(x_r)\right|= \\ &2\|h\|_1.\|e\|_1\end{aligned}$$

But for $\|h\|_1, \|e\|_1 \le 1$, we get $\|h, e\|_1 \le 2$ and:

$$\sup_{h,e\in L_1\|e\|_1,\|h\|_1\le 1}.\sup_{x_r}.\sup_{x_k}\begin{vmatrix}p(x_r) & p(x_k)\\ q(x_r) & q(x_k)\end{vmatrix}.\begin{vmatrix}h(x_r) & h(x_k)\\ e(x_r) & e(x_k)\end{vmatrix}\le$$

$$\|p,q\|_\infty.\|h,e\|_1 \le 2\|p,q\|_\infty$$

This proves that $\|p, q\|_\infty^* \le 2\|p, q\|_\infty$.

Corollary 2.3: If p_m converges in $\|., .\|_\infty$, then it also, converges to p in $\|., .\|_\infty^*$.

Theorem 2.4: If p_m converges in $\|., .\|_\infty^*$, then it also, converges to p in $\|. \|_\infty$.

Proof: Let, p_m a sequence in L_∞ which converges to $p\in L_\infty$ in $\|., .\|_\infty^*$. Then, for any $\epsilon>0$, there exists an $d\in\mathbb{N}$ such that for $m\ge d$, we have:

$$\sup_{x_r}.\sup_{x_k}\begin{vmatrix}p_m(x_r)-p(x_r) & p_m(x_k)-p(x_k)\\ q(x_r) & q(x_k)\end{vmatrix}.\begin{vmatrix}h(x_r) & h(x_k)\\ e(x_r) & e(x_k)\end{vmatrix}<\epsilon$$

for every $q\in L_\infty$ and $e, h\in L_1$ with $\|e\|_1, \|h\|_1 \le 1$. In particular, if we take $q = \{1, 0, 0, ...,\}$, $h = h(x_r)$ with $h(x_r) = \text{sgn}(p_m(x_r)-p(x_r))|p_m(x_r)-p(x_r)|/\|p_m-p\|_\infty$ and $e = \{1, 0, 0, ...,\}$, then we have:

$$\sum_{r=2}^{\infty}\frac{\left|p_m(x_r)-p(x_r)\right|}{\|p_m-p\|_\infty}<\epsilon$$

such that $\|p_m-p\|_\infty \ne 0$. If, we take $q = \{0, 1, 0, 0, ...,\}$, $h = \{h(x_1), 0, 0, ...,\}$ with $h(x_1) = \text{sgn}(p_m(x_1)-p(x_1))|p_m(x_1)-p(x_1)|/\|p_m-p\|_\infty$ and $e = \{0, 1, 0, 0, ...,\}$, then we have $|p_m(x_1)-p(x_1)|/\|p_m-p\|_\infty < \epsilon$. Adding up, we have:

$$\|p_m-p\|_\infty = \sum_{r=1}^{\infty}\frac{\left|p_m(x_r)-p(x_r)\right|}{\|p_m-p\|_\infty}<2\epsilon$$

Theorem 2.5: If p_m converges in $\|., .\|_\infty^*$, then it also, converges (to the same limit) in $\|., .\|_\infty$.

Proof: Since, the convergence in $\|., .\|_\infty^*$ implies that in $\|. \|_\infty$ by Theorem 2.4 and the convergence in $\|. \|_\infty$ implies that in $\|., .\|_\infty$ by Fact 2.1, then the convergence in $\|., .\|_\infty^*$ implies that in $\|., .\|_\infty$.

Corollary 2.6: A sequence is convergent in $\|., .\|_\infty^*$, if and only, if it is convergent (to the same limit) in $\|., .\|_\infty$.

RESULTS AND DISCUSSION

Note 2.7: All these results can be extended to n-normed spaces for any $n\ge 2$. As an extension of Fact 2.2, we have:

Fact 2.8: The inequality $\|p_1, ..., p_n\|_\infty^* \leq n!\ \|p_1, ..., p_n\|_\infty$ holds for every $p_1, ..., p_n \in L_\infty$.

Corollary 2.9: If p_m converges in $\|, ..., \|_\infty$, then it also, converges to p in $\|, ..., \|_\infty^*$. Analogous to Theorem 2.4, we have:

Theorem 2.10: If p_m converges in $\|, ..., \|_\infty^*$ then it also, converges to p in $\|.\|_\infty$.

Proof: Let, p_{1m} be a sequence in L_∞ which converges to $p_1 = \{p_1(x_1), p_1(x_2), ...,\} \in L_\infty$ in $\|, ..., \|_\infty^*$. Then, for any $\in>0$, there exists an $d\in\mathbb{N}$ such that for $m \geq d$, we have:

$$\sup_{x_{r_1}}.\sup_{x_{r_2}},\ldots,\sup_{x_{r_n}}\left|\begin{matrix} p_{1m}\left(x_{r_1}-p_1\left(x_{r_1}\right)\right) & \cdots & p_{1m}\left(x_{r_n}-p_1\left(x_{r_n}\right)\right) \\ \vdots & \ddots & \vdots \\ p_n\left(x_{r_1}\right) & \cdots & p_n\left(x_{r_n}\right)\end{matrix}\right|\left|\begin{matrix} h_1\left(x_{r_1}\right) & \cdots & h_1\left(x_{r_n}\right) \\ \vdots & \ddots & \vdots \\ h_n\left(x_{r_1}\right) & \cdots & h_n\left(x_{r_n}\right)\end{matrix}\right| < \in$$

for every $p_2, ..., p_n \in L_\infty$ and $h_1, ..., h_n \in L_1$ with $\|h_1\|_1, ..., \|h_n\|_1 \leq 1$. Now, take $p_k = h_k = \{0, ..., 0, 1, 0, ...,\}$ for every $k = 2, ..., n$ where 1 is (n+1-kth) term and $h_1 = \{h_1(x_1), h_1(x_2), ...,\} \in L_1$ with $h_1(x_r)$: $= \text{sgn}(p_{1m}(x_r)-p_1(x_r))|p_{1m}(x_r)-p_1(x_r)|/\|p_{1m}-p_1\|_\infty$, then, we have:

$$\sum_{r_1=n}^{\infty}\frac{\left|p_{1m}\left(x_{r_1}\right)-p_1\left(x_{r_1}\right)\right|}{\left\|p_{1m}-p_1\right\|_\infty} < \in$$

Next, if we take $p_k = h_k\ \{0, ..., 0, 1, 0, ...,\}$ for every $k = 2, ..., n$ where, 1 is kth term and h_1: $= \{h_1(x_1), 0, 0, ...,\}$ with $h_1(x_r) = \text{sgn}(p_{1m}(x_1)-p_1(x_1))|p_{1m}(x_1)-p_1(x_1)|/\|p_{1m}-p_1\|_\infty$, then we have:

$$\frac{\left|p_{1m}\left(x_1\right)-p_1\left(x_1\right)\right|}{\left\|p_{1m}-p_1\right\|_\infty} < \in$$

Similarly, if we alter the position of the entry 1 in p_k and h_k for $k = 2, ..., n$ and change the nonzero entry of h_1 accordingly, then we can get:

$$\frac{\left|p_{1m}\left(x_2\right)-p_1\left(x_2\right)\right|}{\left\|p_{1m}-p_1\right\|_\infty} < \in$$

And so on, until:

$$\frac{\left|p_{1m}\left(x_{n-1}\right)-p_1\left(x_{n-1}\right)\right|}{\left\|p_{1m}-p_1\right\|_\infty} < \in$$

Adding up, we get:

$$\left\|p_{1m}-p_1\right\|_\infty = \sum_{r_1=1}^{\infty}\frac{\left|p_{1m}\left(x_{r1}\right)-p_1\left(x_{r1}\right)\right|}{\left\|p_{1m}-p_1\right\|_\infty} < n\in$$

Corollary 2.11: A sequence is convergent in $\|, ..., \|_\infty^*$, if and only, if it is convergent in $\|, ..., \|_\infty$.

CONCLUSION

We can relate the convergence in terms of Gunawan's norm and the convergence in terms of Gahler's norm by equivalence or they are the same convergence for sequences of continuous function.

ACKNOWLEDGEMENT

We would like to express our thanks and deep gratitude to professor Eman Samir Bhaya, for her encouragement, support and invaluable guidance during this research.

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